Quantum Physics 1 - Test 1

A particle of mass m is in the state

$$\Psi = Axe^{-a(mx^2/\hbar + 3it)},$$

where a is some positive real constant and $A = 2\left(\frac{2ma}{\pi\hbar}\right)^{1/4}\sqrt{\frac{ma}{\hbar}}$.

a. For what potential energy $V(x)$ does Ψ satisfy the Schrödinger equation? (*5 points*)
Hint: the Schrödinger equation is given by $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$

b. What are the expectation values $\langle x\rangle$ and $\langle p\rangle$? Explain your answer. (*4 points*)
Hint: you do not have to work out any integrals.

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Quantum Physics 1 - Test 2

Consider a particle of mass m subject to the harmonic oscillator potential ($V(x) = \frac{1}{2}m\omega^2 x^2$), and assume that, at $t = 0$, the particle is in the state

$$\Psi(x,0) = \frac{1}{\sqrt{2}} \left(\psi_0(x) + \psi_1(x)\right).$$

a. (*1 pts*) Add the time dependence to Ψ (i.e., find an expression for $\Psi(x,t)$).

b. (*3 pts*) Calculate $\langle x \rangle$. (*Hint*: use orthonormality to evaluate integrals, and consult the hints at the bottom.)

c. (*3 pts*) Calculate $\langle x^2 \rangle$ and σ_x.

d. (*2 pts*) Is there a moment in time when the momentum standard deviation σ_p can be zero? Explain your answer.

Hint: x and p can be written in terms of ladder operators as follows:

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a_+ + a_-\right), \; p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-).$$

We also have the following ladder operator relations: $a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$, $a_-\psi_n = \sqrt{n}\,\psi_{n-1}$.

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Quantum Physics 1 - Test 3

Consider the *moving* delta-function well, whose potential and normalized solution to the time-dependent Schrödinger equation are given by

$$V(x,t) = -\alpha\delta(x - vt)$$
$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar}\exp\!\big(-m\alpha|x - vt|/\hbar^2\big)\exp\!\Big(-i\big[\big(E + \tfrac{1}{2}mv^2\big)t - mvx\big]/\hbar\Big)$$

where v is the (constant) velocity of the well, $\alpha > 0$, $\exp(a) = e^a$ and $E = -m\alpha^2/2\hbar^2$.

a. *(3p)* Calculate $\langle x\rangle$. *Hint*: You can use the integral $\int_0^\infty x^n e^{-ax}\,\mathrm{d}x = \frac{n!}{a^{n+1}}$.

b. *(2p)* What is the probability to find the particle to either side of the well?

c. *(1p)* Calculate $\langle p\rangle$.

d. *(3p)* Recall the equation for the probability current

$$J \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right).$$

Calculate the probability current for this wave function. Which direction does the probability current flow? *(1p) Bonus:* Express J in terms of Ψ
Hint: To avoid having to differentiate the absolute value, write $\Psi = cf(x,t)g(x,t)$ with each f, g containing one exponent, such that $f = f^*$ and write out J before doing any differentiation.

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Quantum Physics test 4

Consider the following potential: $V = 0$ for $x < 0$ and $V = -V_c$ for $x > 0$, where V_c is a positive constant. For a particle moving to the right with energy $E_0 > 0$:

1. Sketch the potential and write the solution of the eigenvalue equation $\hat{H}\psi = E\psi$ for $x < 0$ and $x > 0$, considering no incoming wave from the right (3 points).

2. Find the reflection coefficient R in terms of E_0 and V_c (3 points).

3. Verify that R and T (transmission coefficient) sum up to 1. T is given by the formula

$$T = \sqrt{\frac{E_0 + V_c}{E_0}} \frac{|F|^2}{|A|^2},$$

with A and F the coefficients of the plane wave travelling to the right for $x < 0$ and $x > 0$ accordingly (4 points).

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Test 5 Quantum Physics 1

a) Using the formula:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2, \tag{1}$$

show that $\sigma_H \sigma_p \geq \frac{\hbar}{2} \langle \frac{dV}{dx} \rangle$. (3 points)

b) Consider the harmonic oscillator ($V(x) = \frac{1}{2} m\omega^2 x^2$). Does the uncertainty relation derived above give you information about the ground state? What does it say about the excited states?

c) A generic state of the harmonic oscillator is a superposition of the ground state and all the excited states. What does the uncertainty relation tell you about generic states? (3 points)

Test 5 Quantum Physics 1

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$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2, \tag{2}$$

show that $\sigma_H \sigma_p \geq \frac{\hbar}{2} \langle \frac{dV}{dx} \rangle$. (3 points)

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Quantum Physics 1 - Test 6

Consider the ground state of hydrogen, of which the wave function is given by

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}.$$

a) *(4p)* Calculate $\langle V \rangle$. *Hint:* Use the fact that $\langle V \rangle = -\frac{\hbar^2}{ma}\left\langle \frac{1}{r} \right\rangle$, and use integration by parts.

b) *(3p)* Using your result of (a), calculate $\langle \mathbf{p}^2 \rangle$. If you did not manage to complete (a), find $\langle \mathbf{p}^2 \rangle$ in terms of $\langle V \rangle$. The energy of the ground state ψ_{100} is given by

$$E_1 = -\frac{\hbar^2}{2ma^2}.$$

c) *(2p)* What is $\langle {p_x}^2 \rangle$? *Hint:* You do not have to do any long calculations.

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Quantum Physics 1 - Test 7

Two particles, each of mass m, are attached to the ends of a massless rigid rod of length a. This system (called a *rigid rotor*) is free to rotate in all three dimensions about the fixed center point. The classical energy of this system is given by $E = L^2/(2I)$, where $I = ma^2/2$ is the moment of inertia of the system.

Now, we consider the case that a is very small and hence we will describe this system quantum mechanically. Thus, we will use operators, resulting in the Hamiltonian below:

$$\hat{H} = \frac{\hat{L}^2}{2I}.$$

(a) (*2 pts*) Find the allowed energies of the rigid rotor (i.e., find the eigenvalues of $\hat{H}$).

(b) (*2 pts*) Find the corresponding degeneracies.

(c) (*3 pts*) Construct the ground state of the rigid rotor.

(d) (*2 pts*) Is there a difference between the classical ground state (i.e., the lowest possible value for the classical energy) and the quantum mechanical ground state energy? Explain how this is compatible with Heisenberg's uncertainty principle.

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Test 8 Quantum Physics 1

a) Consider a system of two non-interacting identical particles, one in state ψ_a and one in state ψ_b. These states are orthogonal and normalized. Show that:

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2$$

where $\langle x \rangle_{ab} = \int x\psi_a^\star(x)\psi_b(x)dx$. This term indicates the *exchange force*. It takes the upper sign in $\mp$ for bosons and the lower for fermions.

b) Consider two non-interacting identical *bosons* in the one-dimensional harmonic oscillator potential, $V = \frac{1}{2}m\omega^2 x^2$. The ground state for a single particle in this potential is:

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Is there an exchange force in the ground state of the two-particle system? If so, is it attractive or repulsive? Assume the particles are in the same spin state.

c) Consider two non-interacting identical *fermions* in the same potential. However, this time the particles are in the singlet state where the total spin is equal to zero. Is there an exchange force in the ground state of this system? If so, is it attractive or repulsive?